Belief Formation Function: A Method to Model Equilibrium Selection in Games

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Abstract

In many game-theoretic models, it is common to see multiple equilibria. There are extensive literature on identifying which equilibrium is more likely to occur, arguments like the focal points, evolutionary convergence, learning and Et cetera. But all of them have certain limitations. In this paper, I propose a new setup that enables us to model the equilibrium selection process. Instead of players best responding to each other's actual strategies, I assume that players best respond to their beliefs about the opponents' strategy. On the other hand, the belief is generated from a belief formation function that may take any observables and map that into a specific belief about the opponent's strategy. This way, we may construct models describing how payoff irrelevant signals and off equilibrium play may shape future outcomes and choosing equilibrium. Then I introduce the outer games to model how exogenous shocks to an inner game are generated. I allow players in the outer game to affect the game structure and belief formation signals of the inner game and thus control equilibrium outcome exogenously. I use several examples to illustrate this new setup and how it is different from more conventional sequential games. I finish this paper by discussing further regulatory assumptions on the belief formation functions.

1 Introduction

Equilibrium selection has been the interest of game theory for a long time. However, modeling the equilibrium selection process is never the easiest thing to do. In this paper, I propose a new formulation of the game, allowing us to model equilibrium selection. I will start with a motivating example.

Assume there are two volunteers recruited to participate in a one-round lab experiment. Both players simultaneously choose a side of the coin. If their choices match, then both of them get 100 dollars; however, if their choices fail to match, then they get nothing. There are two pure Nash equilibria: both players choose the head, or both players choose the tail, and there is one mixed strategy Nash equilibrium: Both players choose the head with probability $\frac{1}{2}$. This is a coordination game. However, if we do not allow players to talk before the play starts, it is unclear which equilibrium they would choose. In reality, we may observe many cases in which players fail to coordinate. Also, we observe that players are more likely to select head than tail. We may explain such phenomenon by arguing that the head is the focal point. But there are still questions about what makes up a focal point and why the head is the focal point, but the tail is not. Lastly, consider the following case. If the players fail to coordinate initially, what would they choose if researchers ask them to play the game again, unexpectedly? Since the second round of the game is played unexpectedly, the participants could not initially have a complete strategy for the two-stage game. Each player has to guess if her opponent will switch action to match her own or if her opponent does not switch and instead, expecting her to switch. It is not easy to construct such a model without introducing some additional elements to a game.

To answer these questions, We need something to describe how each player forms their beliefs when they cannot observe the opponent's exact strategy. Most equilibrium concept in game theory is static concepts and can be interpreted as: if players are assigned an equilibrium strategy, and the strategy profile is common knowledge, then the outcome is stable in the sense that no player would have any incentive to deviate from the assigned strategy. But most equilibrium concepts have very little to say when players do not have enough information about each other's strategy or when people start with off equilibrium play. It remains to explore how players' beliefs change and how the game would converge to an equilibrium.

There are, in general, two cases when we have the equilibrium selection problem. First, players face a new game, but they cannot communicate and form common knowledge of each other's strategy. Second, an existing game may face an unexpected shock on the game structure, so the player's original strategy becomes incomplete. In both cases, players have to believe how each other would play in the new situation, which gives rise to the equilibrium selection problem.

To model the equilibrium selection problem, I follow the idea of O. and W. (1995) that a focal point is formed because players would try to predict how the other players act. They include priors of what the others would do and then best respond to those priors. Their work leads to the level-n theory, which means players have different levels of bounded rationality, and thus experiments may show off-equilibrium plays in lab experiments. In this paper, I extend their idea of "a player has prior of the other's behavior," and I allow more general belief formation functions than the level-n theory. We can model the equilibrium selection process in various games and settings by specifying the belief formation function.

Thus a natural next step is to model why there are new games or why there will be exogenous shocks to an existing game structure. I introduce a multi-level game setup that includes an inner game and a corresponding outer game to deal with it. Assume the players in the outer game has methods to change the game structures or belief in the inner game, and they get different payoffs from different outcomes of the inner game. Then we can use the outer game to study the origin of exogenous shocks.

This paper also touches on what is "exogenous" and "endogenous" to the players. If we construct a game of life for everyone and assume the players are infinite rational, and sophisticated. Then naturally, it follows they can have a correct expectation of every action they choose in the game, in the future, or the past. However, it is never realistic to make such an assumption. No one is such an Omnipotent "god." For every game, there is a border between what is endogenous and what is exogenous. Every player only chooses the action of an exogenously given game structure, and when such structure changes, all the original strategies becomes incomplete. Thus, belief formation functions are required when the players are shocked into a new game. The belief formation function serves as a bridge connecting the old and new game. For instance, there can be a "convention" among players that all believe that everyone else would choose an equilibrium in the new game that most resembles the old equilibrium they played before the shock. There can also be someone who disregards all the previous information and believe that each new equilibrium is played with equal probability. In coordination games, the latter belief formation function may be more efficient and thus preserved by the evolutionary process. But the evolution of belief formation function is not the focus of the current paper.

Equilibrium selection has been the interest of economics theorists for a long time. There has not been an equilibrium concept that can guarantee the existence and uniqueness of equilibrium in all games. When there are multiple equilibria in a game, it is unclear how players should select one among them. To deal with the problem, the first branch of literature starts from focal point theory by Schelling (1960), in which he argues that people would be able to know which is a salient equilibrium according to some payoff irreverent signals. However, it is sometimes unclear what makes a focal point salient (Schelling (1960)), (Camerer et al. (2004)), (Bacharach 1999), (Mehta et al. (1994)), (Chen & Chen 2011), (Bohnet and Cooter (2003)) There are also literature on Evolutionary process to choose equilibrium (Smith (1979)),(Young (1993)), (Nax and Newton (2019)) and learning process are more concerned about long run outcome and have limited predictive power for short run behavior. (Binmore and Samuelson (1997)), (Blume (1993), Blume (2003)). Though prediction in the long run is important to describe well established social structures or conventions, they do not provide sharp predictions on individual's short run behavior when they are presented with an unprecedented situation or when there are exogenous disturbance to a steady system. Lastly, ideas similar to belief formation function has been used in the literature. Forward induction use concepts such as strong belief for rationality to rule undesired equilibria. Such belief, on the other hand, is equivalent to a belief formation function that responds to observed off-equilibrium play (Battigalli and Siniscalchi (2002)),(Monderer and Shapley (1996)). But these arguments are also hard to be applied to different games.

In section 2, I formally define the inner game, outer game, and belief formation functions. Section 3 provides several examples of using the inner game and outer game to model equilibrium selection. Sections 4 discusses regulatory assumptions on the belief formation functions so that they are not chosen arbitrarily. Section 5 briefly talks about the future research agenda, and finally, section 6 concludes the paper.

2 Model setup

To keep this paper succinct, I will first set up the model as a Bayesian game, and the set of actions for each player to be finite. It is not hard to generalize this set of notations to a broader set of games, including sequential games or continuous action set. I first define a general notion of inner game and then build the outer game following the notation of Kajii and Morris (1997).

2.1 Inner Game

An inner game is a collection of $G^I = \{N, \Omega, p, \{A_i, u_i, T_i, \tau_i\}_{i \in N}, F, O, \Theta, X\}$. Here, $N = \{1, ..., n\}$ is the set of players from 1 to n. Ω is the set of states of nature. A_i is the set of actions for player i. Let A denote the action set for all the players. A is the Cartesian product of each player's action $A = A_1 \times A_2 \times ... \times A_N$. . T_i is the set of types for player i, which depends on the state of the world. Given the state, the type of player i is determined by the function $\tau_i : \Omega \to T_i$. The type of a player may affect his or her payoff. Thus we have the following payoff function: $u_i : T_i \times A \to \mathbb{R}$. Finally, we have p as a prior probability distribution over Ω.

A pure strategy for player i is a function $s_i : T_i \to A_i$. A mixed strategy for player i is a function $\sigma_i : T_i \to \Delta A_i$, where ΔA_i is the set of all probability distributions on A_i. Denote A_{-i} (σ_{-i}) as a bundle of pure actions (mixed actions) for all the players except that of player i. Denote the set of all strategy for a player i as Φ_i and let $\Phi = \{\Phi_1, ..., \Phi_n\}$ be the set of strategy for all the players.

We also need some notations to specify the beliefs of a player. A belief of a player i is a probability distribution on the opponents set $b_i : \sigma_{-i} \times T_i \to [0, 1]$ such that $\sum_{\sigma_{-i}} b_i(A_{-i}|T_i) = 1$ for all $i \in N$ and T_i.

For most parts, an inner game is just as any standard game in the literature. The only difference is that when talking about an inner game, we need to specify the observability correspondence O, the belief formation functions F of each player: F_i, a set of tie breaking functions ,Θ and a set of game irreverent signals X.

Let X be a set of game-irrelevant signals. This set of signals is exogenously given, and players in the inner game do not have ways to change it. This set X may include things like the weather of the day, a random number generator, a colored grid's location on a game table, and others. More importantly X includes a set of signals about whether a player is rational or not.

$O_I(h)$ is a correspondence that maps a history to a set of opponents' characteristics and game irrelevant signals $O_i(h) : h \to 2^{\{F,O,\Theta,X\} \cup \Phi}$. I refer to $O_i(h)$ as the observable set to the player i at history h.

Since we are currently focusing on a simultaneous move game, there is only one history before each player chooses their actions. This set can include anything, but this set matters because the player would use what is observable to him or her as input to the belief formation function, and thus his or her belief on the opponent's strategy may change accordingly. Denote $O = \{O_1, O_2, ..., O_n\}$ as the set of observability correspondence for each player.

Assumption: If an element is in the observable set to a player at history h, then the player knows the true state of the element , and the player assigns probability 1 to that state of the element being happening. For instance, if the strategy set is observable, the player would directly respond to the observed strategy set. To represent a common knowledge of an event, we need to put this event into every player's observation set, every player's observation set is also in every player's observation set, the observation set that contains everyone's observation set is also in everyone's observation set, and this recursion shall continue to infinity.

In a classical Bayesian Nash equilibrium, it is commonly assumed that each player's strategy is common knowledge. In other words, each player knows each other's strategy for sure, and everyone knows that everyone knows this fact and until infinite recursion. However, in this paper, I will relax this assumption and substitute it with the belief formation function. Also, to model off-equilibrium play, I relax the assumption of fully rational agents and common knowledge of rationality. So some players can believe that some opponents are not rational and choose strictly dominated strategies.

It is also commonly assumed that each individual's rationality, the game structure are also common knowledge. I will keep such assumptions because otherwise, players would not have sufficient information to make decisions.

For sequential games, each player's history of play and the state of the world may be imperfectly observed. However, in this paper, I will manipulate the observable set. Most importantly, I will remove the assumption that strategies are common knowledge, but instead, beliefs about opponents' strategies are generated by each player's belief formation function. The belief formation function maps from the set of observables of player i at each history h to a unique belief b_i on the opponent's strategy: $f_i : O_i(h) \rightarrow b_i$.

Then, we can define rationality of a player i as: given the player's belief about the opponent's strategy in each state of the world, the player i choose a mixed strategy σ_i such that:

$$E_\Omega[u_i(\sigma_i(\omega), b_i(\omega)|\omega \in \Omega)] \geq E_\Omega[u_i(\widetilde{\sigma_i}(\omega), b_i(\omega)|\omega \in \Omega)] \tag{1}$$

for all $\widetilde{\sigma_i} \in \Phi_i$. It means the player choose the strategy to best response to his belief about what the opponents would do in each state of the world. I model players best respond to his or her belief of the opponent's strategy instead of the actual strategy of the opponents. The believes are generated by the belief formation functions.

If a player is irrational, then I assume such a player uses an exogenously given "strategy formation function" to choose the strategy. A strategy formation function maps from an observable set to a specific mixed strategy. However, I do not include the strategy formation function as a part of the, because there is an equivalent specification using the tie-breaking function.

The last element $\theta_i \in \Theta$ is a tie-breaking function. $\theta : b_i(\omega) \rightarrow \sigma_i(\omega)$, for all b_i and $\omega \in \Omega$. The tie-breaking function is used when, given the belief, a player has two or more pure strategies that generate the same utility. In such a case, rationality does not pin down a specific action profile for the player. In such cases, the tie-breaking function points to a specific mixed strategy for the player to follow.

To model an irrational player, we can equivalently assume that the player does not observe his or her true payoff function, but instead, such player is indifferent between any outcome. In such a case, the player uses the tie-breaking function to make decisions, and that is equivalent to specifying the strategy formation functions.

We need the belief formation function to replace the common knowledge assumption for the following reasons: 1. Players can only best respond to his or her belief or conjectures of the opponents. When we try to put model prediction to reality, people usually have limited information on what the others would do, and thus the outcome may differ from the model prediction. Allowing the discrepancy between the belief and the actual opponent strategy is the missing point. Depend on the model and the equilibrium concept that we use, We can put restrictions on the beliefs such as consistency in the Bayesian equilibrium, belief has to reflect the actual strategy of the opponents, and so on. 2. Introducing the belief formation function enables us to formalize equilibrium selection processes such as focal point, competitions between focal points, off equilibrium play (travelers dilemma), exogenous shock on game structures, and even more complicated general learning process. The equilibrium selection process and exogenous shocks on the game structure are the focus of this paper.

Theorem: Given a game, for each existing equilibrium concepts (EC) that assume rational players, there exists a set of belief formation functions F, tie-breaking functions, and observability correspondence O which can replicate each of the equilibrium strategies.

Sketch of proof: 1. Construct observable correspondence such that each player's action is in the observable set in all the history. Assume all players are rational. Then construct the observable set according to the game's specification and the equilibrium concept (usually assume that each player's strategy is common knowledge).

2. Define the set of equilibrium strategy profile with respect to an equilibrium concept (EC) in a game as $\Phi(EC, G)$. Let $\sigma_(EC, G) \in \sigma(EC, G)$ as an equilibrium strategy set. Moreover, denote $\sigma_i(EC, G)$ to be the strategy of a single-player i. If there are j elements in the $\Phi(EC, G)$, we can number them from 1 to j.

3. Include a number $c \in [1, 2, ...j]$ in the set of observables for all the players. The number c is observed before the players choose their strategies, and c is determined exogenously. Thus c serves as a coordination device. Each player i has a belief formation function that takes c as an input and then maps the number to her opponents' corresponding equilibrium strategy $sigma_{-i}(EC, G, c)$. This notation means the c_{th} equilibrium strategy of the opponents when the game is G ad the equilibrium is EC. (here is a slight abuse of notation.) Assume the belief formation functions and the tie-breaking functions are common knowledge, and all of them makes consistent predictions. By consistency, I mean players would choose the strategy that all the opponents predict them to do, given their rationality and tie-breaking functions.

4. Then, the player chooses an action to maximize the expected payoff given his or her observables and belief formation function. There might be cases in which a player has multiple strategies as the best response to a belief of the opponents, but only one of her own strategies is the equilibrium strategy. Thus, construct the tie-breaking function for each player always to choose his or her equilibrium strategy.

This is a simple but important theorem. For most of the models that do not study off-equilibrium play, we want to put more restrictions on the structure of the belief formation functions; otherwise, almost any outcome can be justified by some ad-hock belief formation functions. Most of the structure on the belief formation comes from equilibrium concepts. In some sense, we want to restrict the players to play the outcomes that correspond to some well-established equilibrium concepts. This theorem tells us how to construct such belief formation functions, observability sets, and tie-breaking rules. However, such a rule may not be followed when we try to write down models for off-equilibrium play.

There are also different specifications of those elements of an inner game to guarantee an equilibrium play. Carefully write down those conditions forces us to meditate on the necessary or sufficient conditions needed to guarantee Nash equilibrium. We shall quickly see how strong the assumptions are (such as common knowledge of strategy) if they were met in reality. Such assumptions may be satisfied if players are allowed extensive communication. However, it is not appropriate for most single-shot games that start without a sufficiently long communication stage.

2.2 Outer game

To complete a model for equilibrium selection or exogenous shocks to an inner game, we need to construct an outer game. The formal representation of the outer game is the following:

$$G_{out} = \{M, \Omega^{out}, p^{out}, G_{in}, \{A_i^{out}, u_i^{out}, T_i^{out}, \tau_i^{out}\}_{i \in M}, F^{out}, O^{out}, \Theta^{out}, X^{out}\}$$

Also, for simplicity, this paper's outer game is defined as a finite strategy Bayesian game with the additional elements of $F^{out}, O^{out}, \Theta^{out}, X^{out}$. For notations with the same definition as the inner game, I add a superscript "out" (or sometimes subscripts) to denote that they are an outer game's elements. M is the set of players. I will refer to the players in an outer game as external influencers or exogenous players because they have methods to affect the structure or the outcome of an inner game.

An outer game G_{out} is always defined with respect to an inner game G_{in}. But the most important thing about an outer game is that: the outcome of the outer game can change the game structure of the inner game, such as the available strategy, payoff, and other signals. Thus, I will represent the game structure of the inner game to depend on the outcome of the outer game: $G_{in}(A^{out})$, here A^{out} is the set of actions chosen by the players in the outer game.

I also allow the outcome of the inner game to affect the outer game players' payoff, let A_{in} denote the actions chosen by the players in the inner game, so we have the payoff functions $u_j^{out} : \Phi_{in} \times \Phi_{out} \to \mathbb{R}$ for all $j \in M$. To better understand this setup, you can think about external influencers as lawmakers. They can designate specific activities as punishable and change the payoff to structure to the citizens. Consequently, such change could modify the equilibrium in the society. On the other hand, the lawmakers derive a different level of utility from different equilibria. Some of them might care about improve social welfare; some others might have private interests to achieve. The game among the lawmakers can thus be modeled in the outer game.

I always assume that $\{M, \Omega^{out}, p^{out}, G_{in}, \{A_i^{out}, u_i^{out}, T_i^{out}, \tau_i^{out}\}_{i \in M}\}$ is common knowledge to all exogenous players. Everything in the inner game is observable to the exogenous players, include the endogenous player's belief formation function, observability function, tie breaking function and signal sets. And for each player $j \in M$, $\{F_j^{out}, O_j^{out}, \Theta_j^{out}, X_j^{out}\}\}$ is observable, because, otherwise the players would not be able to make decisions.

We can also set up the multi-level outer game: the outer game can have a higher level outer game. For instance, the lawmakers' payoff from passing different laws may depend on the game among those interest groups that he or she serves. Theoretically, we can trace up the multi-level outer game, and eventually, we can have a system of games that includes every human and every strategy. Till that point, the only outer game is the move of nature.

However, for a specific model, it is not always necessary to have an outer game. If an inner game's belief formation function does not take any exogenously designed signal as inputs or there are no exogenous shocks, then the outer game is unnecessary. In such a case, we can construct models such that the outer game is null: there is no player or decision making process in the outer game, so the outer game's outcome does not change the inner game. Also, the outer game is not always necessary if we only want to explain equilibrium selection. Specifying F and O in the inner game would sometimes be sufficient.

I should also point out that an outer game is fundamentally different from an additional stage of subgame. We can set up repeated outer games: First, some law makers play an outer game that passes a new law. Then upon observing the law, the general public converges to a new equilibrium given the law. However, while doing so, the general public does not anticipate what equilibrium they choose to impact the future law-making process. They think the law is final. However, if the new equilibrium is not desirable to the lawmaker, there can be a second and third round of the outer game and modifying the law. Each time the law changes, it imposes an exogenous shock to the inner game that people play. However, if we formulate this repeated law-making process as a sub-game, the players in the inner game, which is the general public, do anticipate that their action would reversely affect later lawmakers' decisions. Outer games are used to model exogenous shocks and how equilibrium would jump before and after such shock.

The line between using an outer game or adding some additional stages to a subgame is not clear. Using the outer game is in general, a simplification because we can then ignore the anticipation effect of players in the inner game. It is usually subject to argue whether such simplification is appropriate. To get some intuition, We can consider the following cases:

1. Subjects in a game theory experiment vs. Researchers who design the experiment. The subjects come to the lab, and the researchers tell them the experiment's rule and payoff. The settings of the experiment are considered as "exogenous" by the subjects. It is unlikely that the participants can anticipate how their choice may affect the future experiment design and manipulate strategy to maximize the joint payoff in current and future experiments.

2. The researchers vs. Editors of an academic journal. We might also model how the payoff to the researchers is established. If the editors and referees agree that the paper is good enough to be published, the authors can get one payoff unit. However, it might not be appropriate to model

journal editors' decision-making process as an outer game that determines the researcher's payoff structure. Since the authors can defend their paper and change the publishers' minds, they should be model in one game.

3. Citizens vs. Lawmakers of a country. For most of the time, the general public would take the law as given. In such cases, it is appropriate to model the people's game as an inner game while modeling the lawmakers' game as an outer game. However, there are some cases when the new law becomes very controversial. Thus people might protest against the law in the hope of changing it. In such a case, it is no longer appropriate to separate the lawmaker and the general public into the inner game and outer games.

There are two main reasons why it is essential to model exogenous shocks using outer games. We human beings are not infinite sophisticated as decision-makers. There are always things that we fail to anticipate when making decisions. We take so many things as exogenously given when we make decisions, such as the law, some national policies, people's preferences, social norms, conventions, and so on. However, these things are subject to changes. Whenever the changes occur, people's strategy in the original game does not apply anymore, so in this paper, I will use the belief formation function and outer games to formulate the equilibrium selection process.

Another critical difference between the conventional specification of dynamic games and the inner-outer game in this paper is that: having a different stage in the subgame does not solve the equilibrium selection problem. If there are multiple equilibria in the dynamic game, we still need to impose additional assumptions on which equilibrium would be chosen.

3 Examples

In this section, I will show how the new game setup works using three examples. The player, action, and payoff of the inner game will be the same, but the belief formation function, observable set, and outer game vary case by case. I use these examples to illustrate: 1. equilibrium selection without exogenous shocks. 2. exogenous shocks on payoff irrelevant signals and equilibrium selection. 3. exogenous shocks on payoff functions and equilibrium selection.

First, I will set up the players, action set, payoff functions, and inner game timing. I call the game: "Battle of a Party." It is a three players version of the "Battle of sexes." See the figure 1 for the payoff table. Think about three friends who would always hang out for a party each Friday night, but they never specify where to meet. For simplicity, I will call them players 1, 2, and 3. There are three bars that they can go to: A, B, and C.

All the friends want to party together. So each gets a payoff equals to 2 if all of them meet at the same place. If only two people are in the same place, each gets a payoff of 1, and the person who alone gets zero payoff. If all of them go to separate places, then all get zero utility.

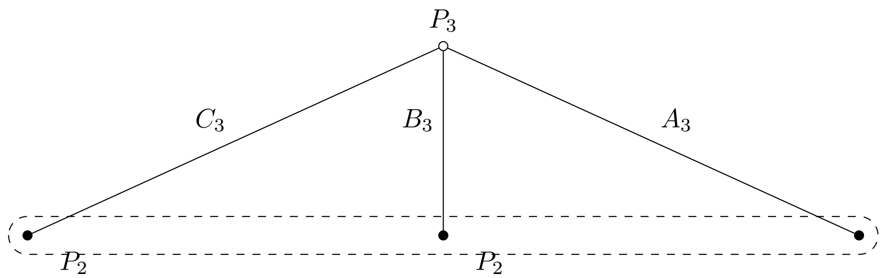
Also, just as in the battle of sexes, each player has his favorite place. Assume player one likes A, player two likes B, and player three likes C. A player gets one additional unit of utility if he goes to his favorite place and at least one of his friends is there. In other cases, if a player is alone, he gets zero utility no matter where he goes.

Then we can write down the game as is figure 1. Let player 1 chooses the row; player 2 chooses the column, and player 3 chooses the payoff table (So there are three tables). If player 3 goes to place A, then the payoff is in the upper left table; if player 3 chooses B, then the payoff is on the upper right table; and if he chooses C, then the payoff is in the bottom left.

We should allow mixed action. To denote the probability of player $i \in \{1, 2, 3\}$ going to A is p_i; the probability of going to B is q_i and the prob of going to C is $1 - p_i - q_i$.

Assume all players have a standard δ discounted utility, and players are rational. Each player chooses a strategy to maximize his or her expected utility given their belief about the other player's strategy and their tie-breaking function.

In this model, the game structure and the rationality of players are common knowledge to everyone. They take these elements as given and choose a strategy to best respond to their belief about their opponent's strategy. Assume the original game is in a subgame perfect Nash equilibria,



C_3:

P_1 \ P_2	A_2	B_2	C_2
A_1	2,1,0	0,0,0	0,1,2
B_1	0,0,0	1,2,0	0,1,2
C_1	2,0,2	1,0,2	2,2,3

B_3:

P_1 \ P_2	A_2	B_2	C_2
A_1	2,1,0	0,2,1	0,0,0
B_1	1,0,1	2,3,2	1,0,1
C_1	0,0,0	0,2,1	1,1,0

A_3:

P_1 \ P_2	A_2	B_2	C_2
A_1	3,2,2	2,0,1	2,0,1
B_1	0,1,1	1,2,0	0,0,0
C_1	0,1,1	0,0,0	1,1,0

Figure 1: Three player "Battle of the Party" game

in which everyone goes to bar A for every period, and this game is common knowledge among players. However, one day, the game structure changes unexpectedly. So no one had any belief or strategy prepared for the new game. To each player, such change is exogenously given. Later, we can discuss the strategic interactions between the bar owners, which is a higher-level game. The equilibrium result of such higher-order games would determine the structure of a lower order game (in this example, it is the game that the consumers play).

Assume that on the day t_0, bar B unexpectedly announces a coupon to attract more customers. The coupon says: from today to the infinite future, for anyone who holds this coupon, come to my bar, you can have two drinks for free. Assume this coupon is worth four units of utility to each player. So if a person gets it, he will go to the bar B no matter what the others do. We will continue to assume the person who receives the coupon still derives other utility as specified in the previous game. Assume now every player knows the coupon's content, and all the players observe that the coupon is given to player 2. In other words, there is an unexpected shock from the outer game (the game played by the bar owner) that changes the payoff of the inner game. With the coupon, before the start of period t_0, players 1 and 3 observe that player 2 receives the coupon and thus will deviate to bar B for all later periods. We can then check what players 1 and 3 do, depending on their beliefs, since the original strategy does not specify what to do in this new case.

To illustrate how the equilibrium switches, for simplicity, we assume there is no communication among players, so they cannot coordinate their beliefs or strategies. I do not assume common knowledge of the belief formation function or each player's actual strategy after the exogenous shock. So we can see how different belief formation functions affect the new equilibrium. In the following sections, I provide several examples of belief formation functions.

3.1 New Equilibrium Strategy and Belief Formation Functions:

In the following discussion, I will ignore the belief formation of player 2 because for all her possible beliefs about the other two players, she always chooses to go to bar B. So her belief formation function is irrelevant. Given that they observe player 2 switched due to the coupon, whether player 1 and 3 would switch depends on their belief of each other's action after observing the deviation.

Belief Formation Function 1: No switching If player 3 has a belief formation function that maps the change in the game structure to a new belief that player 1 will stick with bar A for all periods, then no matter the player 3 chooses A or B in the later periods, he gets one unit of utility every time. The tie-breaking function for player 3 can be: when there is no incentive to switch out of the bar that he went to in the last period, he stays in the same bar as in the last period). If player 1 also has a similar belief formation function that maps the change in the game structure to the belief that player 3 will remain in A, then he will also remain in A because, in this case, he gets 2 units of utility instead of 1. So, with the above two belief formation functions and tie-breaking rules, player 1 and 3 remain at A, and player 2 goes to B. This outcome also coincides with a subgame perfect equilibrium.

If player 3 has a tie-breaking rule that allows a strictly positive probability of switching bars when he is indifferent, such tie-breaking rule may result in off-equilibrium play. I will discuss such cases later.

Belief Formation Function 2: Switching If player 3 beliefs that player 1 will go to B for all later periods after seeing player 2 switches, player 3 would best respond by also switching. If player 1 also beliefs that player 3 switches, then both of them go to B. This outcome also coincides with a subgame perfect Nash equilibrium.

Belief Formation Function 3: Other beliefs Players do not need to always settle down in a stage game equilibrium. Player 1 and 3 may go to A together in one week and go to B in another week. There could be other patterns overtime depend on their beliefs of each other. A usual assumption in many existing papers is that they will settle down in a Pareto-optimal outcome in the future repeated. This example shows that both players 1 and 3 go to B for all future periods. However, this assumption does not always apply if the game allows multiple Pareto-optimal outcomes. In such a case, even negotiation might fail to coordinate strategy and beliefs. For instance, we see experimental papers that find players fail to coordinate on the Battle of Sexes, which has two Pareto-optimal equilibria.

We can further discuss the case if the beliefs of players 1 and 3 are not consistent with each other. When we assume that there is no common knowledge of belief formation function, inconsistent belief may occur and result in off equilibrium play. There will be further back and forth of switching action based on the individuals' higher-order beliefs. For instance, if player 1 believes that player 3 does not switch, player 3 beliefs that player 1 does switch. Then 3 will go to B, and 1 will remain in A. They would quickly realize that the opponent's action is not consistent with their beliefs, resulting in an off-equilibrium play. When this happens, what players 1 and 3 will do depends on how their belief formation functions respond to off-equilibrium plays. In some extreme cases, they may never end up settle down in one specific bar without communication to coordinate their beliefs. I further discuss more complicated belief formation functions and some regulatory assumptions in section 5 when belief formation functions are not common knowledge.

The above model is a more involved version of the coordination game in the introduction section. The players try to coordinate to maximize payoff. However, due to the unexpected shock and lack of communication, players have to "guess" what their opponents will do based on the belief formation function.

We can see that: the "focal point" is more complicated than just pointing to an equilibrium. The conditional reward or punishment has to be tailored to players' belief of the continuation strategy. A one time offer might not be sufficient to change the equilibrium strategy in the long run. In terms of the law: some laws can alter the equilibrium once, and people's actions will be altered for later periods. Such as the law on which side of the road people should drive on. This coordination is likely to succeed because nobody will expect the side of traffic to switch back after some time. However, for some other law, the equilibrium switching is not a "do it once, and it works forever" type of law—for instance, the law to stop illegal gambling in China. People expect society to return to the original gambling equilibrium once the police effort to control illegal gambling decreases. The underground casinos and gamblers meet "spontaneously" at the old place and old-time after the law enforcement loosens. The following natural step is thus to model how the policymakers decide which rule to announce, and we need to step into the realm of outer games. In the next section, we can take the bar owner's decision into account. The bar owners are just like any exogenous players who can decide the customers' payoff when going to different bars.

3.2 The advertisement game for the bar owners: unexpected signal changes

In the previous model, I assumed that only bar B gives player two a coupon. However, there is no reason to assume that the other bar owners cannot give away coupons or put up advertisements to attract more customers. This subsection will build an outer game that describes the advertisement competition among the bar owners. The advertisement only serves as a signal similar to the "focal point" and does not change each consumer's payoff structure. However, if the advertisement is an input to the belief formation function and thus would affect the equilibrium outcome of the inner game, then the bar owners would have the incentive to compete for the advertisement. In the next subsection, I study how the bar owners may compete to offer the coupon.

Assume that initially, all the customers go to bar A as specified before. Their initial beliefs are: each player knows that the other players will go to bar A for all the periods. The strategy that everyone goes to bar A is common knowledge. Assume day t_0 is a special holiday, so the outer game opens up on that day. Each bar owner j can put up an effort $e_j \in [0, \infty)$ to do advertisement. This effort becomes a non-refundable cost to each bar owner. The advertisement does not change the payoff function for each customer. Instead, the advertisement serves as an input to the belief formation function of each player. Assume that all three customers would believe that their friends would go to the bar that puts up the highest advertisement effort for all later periods. Assume that the belief formation function is common knowledge to all the consumers and all bar owners. In this case, all players would go to the bar with the highest advertisement level for all periods after the holiday t_0.

So, the outer game among the bar owners can be formulated as a war of attrition. Assume that if one customer goes to a bar for one period, the bar owner earns one profit unit. Thus at day t_0, the value to a bar owner if all the customers go to her bar for all later periods is $v = \frac{3}{1-r}$ where r is the discount factor of each bar owner. The payoffs are the following: If a bar j gives the highest bid among all, it gets a payoff v. If two or more bars give the same highest advertisement level, then the equilibrium in the inner game would be chosen randomly among those bars with equal probability. The expected winning value becomes $\frac{v}{n}$, where n is the number of bars in the tie. However, all the winners or the losers have to pay for the advertisement cost.

This game has an unique mixed strategy equilibrium, in which the cumulative probability of a bar owner choose an advertisement effort level e_j is

$$p(e_j) = \frac{1}{v} e^{-e_j/v}$$

(Maynard Smith 1976). So each bar owner would choose an advertisement level according to this CDF. Then, the three customers observed the level of effort and the equilibrium if the inner game after the holiday would be determined.

The above example is a competition of focal points. In many cases, there might be multiple signals that can all serve as focal points. With the presence of contradicting focal points, we need to specify belief formation functions to model player's choices. Then the policymakers who have control over those signals may take advantage of it. For instance, the airport managers may put up signs of "meeting points" to help travelers' gathering. However, a book store may also name itself "meeting point book store"; a burger shop may name itself "meeting point burger shop," so it can attract more consumers who happen to set the store as the meeting point. It is hard to model competing for focal points using conventional models, so this paper intends to provide a more organized way to construct such models.

We can also build more involved models to describe the formation of focal points. For instance, think about a repeated coordination game where the players observe many payoff irreverent signals in every round. Those signals are randomly given. Assume that players do not observe each other's belief formation function, and they do not initially map certain signal directly to an equilibrium outcome. Instead, they can keep track of the correlation between each signal and the probability of successful coordination. By sheer chance, some signals will have a higher predictive power of the equilibrium than the others. If all the players learn such a pattern, they may more likely coordinate on that signal, further increasing the signal's prediction power. Eventually, the signal becomes a salient focal point. In the next section, I would like to talk about the outer game that could affect the inner game's payoff.

3.3 Coupon Game of the Bar Owners: unexpected payoff changes

Next, we can consider a different outer game for the bar owners. In this case, they can choose how to distribute coupons on holiday. It is a one-shot simultaneous move game for the bar owners. At the end of the day before the holiday, they simultaneously choose which consumer gets the coupon, which is worth four units of utility to the consumer whenever used in any period. However, because now a player can get the coupon from multiple bars, their action is not determined by the coupon allocation. As soon as the bars give out the coupons, I assume that all the consumers would know the payoff structure in the new game, and they also know who gets the coupon.

Assume all the bar owners are profit maximizers with a discount rate of r. Each of them wants to maximize the current value of the future payoff. Assuming the three customers in our model make decisions independent of all the other consumers who are not modeled, the bar owners can regard these three customers as a small independent market and try to profit within this market. The bar owner's payoff is that: In every period, for each player who goes to a bar j, the owner would earn one unit of profit.

Since the consumers do not expect the coupon before the holiday, their original strategy would be incomplete after the payoff shock. So we need to specify their belief formation function to model what they would do. There are many possible belief formation functions for the players when they encounter an unexpected payoff change. Here I study a relatively simple type of belief formation function, as specified below. I assume all the players have the same function, and it is common knowledge.

1. No payoff shock (no coupon is issued): All players believe that everyone would remain at A for all later periods.

2. With payoff shocks, if there is a single Pareto-optimal equilibrium, then players believe that everyone would go to that equilibrium.

3. With payoff shocks, if there are multiple Pareto-optimal equilibria, each player believes that the opponents choose the equilibrium in which most players go to A. If that does not select a unique equilibrium, they believe that all opponents choose the one with the most number of people going to B. Furthermore, if that does not select a unique equilibrium, they believe that all opponents choose the one with the most people going to C.

4. This criterion guarantees to select a unique equilibrium no matter how the coupons are allocated, so we do not need to specify the tie-breaking function.

Now suppose the only available type of coupon is the one that allows the holder to get two free drinks in each period if used. The coupon will cause a loss of $c = 1.4$ units of profit for the bar owner if one player uses it in a period. Then, each bar owner can choose whom do they want to give the coupon to. Consequently, each owner would have $1 + 3 + 3 + 1$ pure strategies, and thus, there will be 8^3 outcomes to consider. Lastly, assume that bar owners can only offer the coupon on holiday t_0. In essence, the bar owners play a simultaneous move, one period game, and multiple Nash equilibria exist.

To model which outcome would the bar owners choose, I assume a commonly observed coordination device to all the bar owners. The coordination device generates a number that corresponds to each Nash Equilibrium in the outer game, and the bar owners would believe that the other players choose the equilibrium strategy accordingly. Assume the following tie-breaking function for the bar owners: when she is indifferent between multiple strategies given her belief, the player chooses the mix strategy that corresponds to her strategy in the equilibrium. So, to solve the model, we need to solve all the Nash equilibria.

The details of solving the Nash equilibria are in appendix 1. I show that the set of Nash equilibria in the outer game is the following: bar A gives coupons to any two customers. At least one bar B or

C gives coupons to the same two customers as bar A does. And there is one customer who does not get any coupon from any Bar. The inner equilibrium is that all the players go to bar A no matter which equilibrium in the outer game is chosen. Finally, bar A earns a positive profit and B, C earns zero profit. This conclusion applies to all cases in which the cost of coupon $c \in (1, 1.5)$. The outer equilibrium will change if the cost of the coupon is different.

Till this point, the game is complete. We model the decision-making process of the exogenous players and endogenous players. The equilibrium selection process is also taken care of. If we want, we can also build a higher level outer game that models, for instance, the policymakers' decision process regarding rules that regulate the bar promotion.

4 Regulatory Assumptions on Belief Formation Functions

If there are no regulatory conditions on the belief formation function, it is possible to justify any outcome in a model using some ad-hoc belief formation functions. So in this section, I propose two minimum regulatory conditions that should be followed when constructing belief models. They are regulatory conditions that help avoid self-contradicting beliefs or no asymptotically stable outcome.

4.1 First assumption

1. The first assumption is: if the belief formation functions ever become mutually known, then the belief formation functions for each player has to be consistent with each other. Otherwise, a player may have a belief that contradicts what she observes to be true.

For instance, think about a straightforward repeated coordination game, two players each can choose a side of a coin in each round. If both choose the same side, they get a payoff of 1, and otherwise, both get zero. Assume in the first period, the belief formation function is not common knowledge; neither is the rationality of each player. However, before the second period, both the rationality and belief formation functions become common knowledge. Assume both players use the following simple belief formation function: they believe that the opponent would choose the same side of the coin as the opponent did in the last period. Suppose player 1 believes that her opponent would choose head in the first period, and player 2 believes that his opponent would choose tail in the first period. Such belief formation functions are not consistent with each other. Based on their belief in the first period, player 1 chooses head, and player 2 chooses tail. The problem starts in the second period. Based on player 1's belief formation function, she should believe her opponent to play the same action in the first period: tail. However, since she also knows the belief formation function of player 2, she knows that player 2 would believe she still plays head, and thus player 2 would best respond by playing head too. This result of reasoning, however, contradicts her belief formation function. It is unclear how the player would behave in such a case. Thus, when we build a common knowledge model in the belief formation functions, we should build the model such that there is no contradiction.

4.2 Second assumption

2. The second assumption is: if the belief formation functions are not common knowledge, then the belief formation function should result in convergence to an equilibrium outcome with probability one if a stage game repeat until infinity.In reality, players usually do not have common knowledge of each other's belief formation function. They have to predict what the opponents would do. For instance, a max-min utility can be derived from a max-min belief formation function. However, if we do not put any restrictions on the belief formation functions, the model would be too arbitrary, and almost all outcomes are possible with respect to certain belief formation function. So I propose the second assumption above as a minimum requirement for a belief formation function when consistency is not required. If this assumption is violated, players would be stuck in some inefficient outcomes and would never be able to settle down to a "steady state."

Still, take the coin cooperation game as an example. We still assume that both players use myopic best described previously as the belief formation function, and player 1 starts with playing head and player 2 starts with playing tail. Then, each player would choose the side of the coin that the opponent played in the previous round. However, since they started with a different side of the coin, their action would never settle down to successful cooperation, no matter how many rounds they play. It is unreasonable to assume that a person would be unable to see the pattern resulting from the myopic response and yet not modify her belief. If instead, player 1 believes that the probability of player 2 choose the head in the current period is equal to the fraction of time that player 2 chooses the head in all observed history, this belief formation function would guarantee convergence in at most three rounds no matter the initial play in the first period. The proof is in appendix C. To avoid the problem of non-convergence, one easy solution is to assume that all players observe the same information in all the history and have the same belief formation function. Then we can avoid non-convergence caused by the contradiction in beliefs.

To continue in this route, we need to use theoretical work on the general proprieties of the belief formation functions such that players would eventually settle down in an equilibrium. However, also we need to use empirical methods to elicit essential belief formation functions.

4.3 Third assumption

3.Regulation on belief when a player appears both in the inner game and the outer game. There are two main case: 1. the outer game takes place before the inner game. This is the less tricky case. It only requires the player in the outer game correctly anticipate what will herself do in the inner game. 2. the inner game takes place before the inner game. In this case, because the player who also plays in the outer game will anticipate her activity in the outer game, and this is going to cause something like an anticipation and would thus require "subgame perfection", then in that case, the player might not play as her peer anticipated in the inner game. But we should allow such discrepancy in belief.

5 Discussion and Directions of Future Research

The following directions are worth exploring for future projects:

We can first formulate existing equilibrium concepts and selection criteria into some belief formation functions and observable sets. The first point follows from theorem 1. For each equilibrium concept that assumes rational players, we can find a set of outcomes, and for each outcome, we can find at least one equilibrium selection functions and one observable set that would result in the players choose that function. Moreover, we can model the commonly used equilibrium selection process. As the proof in Theorem 1, we can number the equilibria from 1 to n if there are n of them. Then, we can specify another equilibrium selection function $q_i : O_i \to e$ that takes any input in the observable set of player i to a belief corresponding to e. Then we can represent any equilibrium selection process using the composite function:

$$f_i(EC, O_i, q_i(O_i))$$

If q_i leads to the same number in all possible history, then the results must be an equilibrium play. Otherwise, we can have off equilibrium play as demonstrated in previous examples. Writing down the observables and belief formation function conditions is important for thinking through the necessary and sufficient conditions to achieve each equilibrium and equilibrium selection criteria. However, even in very simple games such as the battle of sexes, the belief formation function is large. Do the males want to be gentlemen and thus choose the equilibrium in favor of the lady? Are men more stubborn and selfish, so they would like to choose their favorable equilibrium? These questions lead to the following two research agendas.

Second, we can try to build a theory on the evolutionary basis of the belief formation function. Some belief-formation might converge faster to the equilibrium and thus more efficient; some might result in a higher probability of getting favorable results. Some belief formation functions might have be acquired due to some historical events. The dynamic evolution might depend on the details of the game and context. So doing theoretical work is also important and directly following from the first point.

Lastly, when we face real-world problems, the situation might be so complicated that it is impossible to write down a thorough evolutionary process that pins down a specific belief formation function. That is when doing empirical work to elicit belief formation function is important. In reality, there is rarely common knowledge of the belief formation function, people form their own belief about the other's action, and individuals may disagree with each other. But we can survey people how they form their beliefs, what it takes to persuade them, how they value traditions, and so on.

There is a type of persuasion that is different from Bayesian persuasion. For instance, when a new law serves as a focal device, each individual has to form some beliefs about how many people would follow the new law's new focal point. However, more importantly, such belief is not going to be a fixed number that is constant over time. The belief formation function may depend on other signals such as the observed credibility of law as a focal point device. If a person lives in a society that inexperienced that citizens almost always follow what the law suggests, then based on this experience, a new law would be more likely to be followed by the individuals, which, interns further reinforce people's belief law are credible.

If the law is rarely carried out in another society, then the law's credibility as a coordination device might be undermined. Until most people do not believe that most laws would not be carried out, then the lawmakers would be stuck in a significantly bad situation in which people no longer believe what they try to implement. The lawmakers may need some creative way to re-establish the credibility of the law.

Here I present an interesting story in Ancient China. In 356 B.C., which was in the Warring States period, Shang Yang (390 – 338 B.C.), who served the state Qin as a chancellor, planned for a series of reforms to the state. However, he worried that the general public might question his credibility and thus not following his reform. So one day, be erected a long wooden log in the middle of the capital and announced: whoever can move this log to the city gate shall be rewarded 10 Jin of gold (which is approximately 2.5 kilograms of gold). No one initially believed in him, so Shang Yang raised the prize to 50 Jin gold (approximately 12.5 kilograms of gold). Eventually, someone who was bold enough carried the wood to the city gate, and to everyone's surprise, he gets the reward as promised. After that, Shang Yang earned the reputation as a "man of his words." His reform eventually turned out to be a success. It significantly enriched the state Qin, and eventually, in 221 B.C., Qin defeated all other six rival states and established the first unified dynasty in Chinese history.

It is no longer testable how the wooden log story entered people's belief formation function at that time. However, if we can reproduce similar social experiments, we can model the belief formation process and test it empirically. The findings of such research should have significant policy relevance. With information on the general public's belief formation function, we should be able to estimate the success probability of each policy in shifting equilibrium. We can study what effective ways to establish trust are or how much damage a breach of the oath would do.

6 Conclusion

This paper proposes a new method to model the equilibrium selection process and exogenous shocks to a game. The technique is to specify the belief formation function and the observable set of each player, and when it is necessary, we can define an outer game concerning the inner game of our interests. This new framework can model the formation of either equilibrium outcomes or off-equilibrium outcomes. However, I do not provide any general theorem on equilibrium selection; instead, I offer a new formalization that allows researchers to model such a process.

7 New interpretation of some classical equilibrium concept

In this section, I provide some new interpretation of some classical equilibrium concepts.

7.1 Nash equilibrium

Using the inner game set up, we can interpret a Nash equilibrium as: Before the start of the game, all the players are told the structure of the basic game, and they are assigned a equilibrium strategy, and they are also told the strategy of the other players. In addition, they know the strategy set is a common knowledge. Finally, even if they decide to deviate, there is no way to transmit such decision (the other people do not observe your deviation), and you do not observe other player's deviation either. In this way, if all the players are rational, none of them would have any incentive to deviate from the pre-assigned equilibrium strategy, and in this sense, the equilibrium is stable. The important assumption to sustain an Nash equilibrium is that no player has any way to transmit information about deviation.

7.2 Subgame perfect Nash equilibrium

A subgame perfect Nash equilibrium is needed because we want to allow players to observe deviation: take the firm entry game as an example. There is a incumbent firm and a new entrant. The new entrant moves first and choose whether to enter or not. If the entrant does not enter, it gets a payoff 0, while the incumbent firm gets a payoff equals to 3. If the entrant choose to enter, then the incumbent choose whether to fight or accommodate. If the incumbent choose to accommodate, then both players get payoff equal to 1. If, however, the incumbent choose to fight the new firm, the entrant would get a payoff of -1 and the incumbent would get a payoff equal to 0.

There are two Nash equilibrium in this game, first: the new entrant choose to enter and the incumbent choose to accommodate; second, the entrant choose not to enter and the incumbent choose to fight. However, latter is not a SPNE for the reason that the fight is not a profitable maximizing strategy once the entrant choose to enter. This argument against Nash equilibrium in sequential games seems very intuitive and thus very compelling. However, there is some very assumption about the belief formation function needed to make the argument for SPNE to work.

Assume that currently both players are preassigned the second NE strategy: the entrant do not enter and the incumbent fight. In this case, the deviation of the entrant does not make sense if she does thinks that the incumbent will also deviate. To make multiple deviation allowed, the incumbent has to observe that the entrant has deviated, and the entrant has to know that the incumbent can observe her deviation and would be able to deviate afterwards. If one of those condition is not satisfied, then the logic favoring for the SPNE does not go through.

If the incumbent does not observe the entrant's deviation, then the incumbent does not have incentive to deviate from his originally assigned strategy: fight. Because if the incumbent believes that the entrant will not enter, then no matter what the incumbent does, he always gets 3 units of payoff. Or if the entrant does not know that the incumbent can observe her deviation, then she would believe that the incumbent will stick with the original strategy and thus she also would not deviate.

It is the fact that the first player's action is in the observation set that enables the entrant to deviate and then expect the incumbent to also deviate. Without this critical observability assumption, Nash equilibrium can be sustained. In some sense, the argument for using SPNE is built upon multiple deviation in sequential games.

More deeply, we need to touch the meaning of observability and belief formation function. In this paper, I assumed that when ever a thing is observed to a player, then the player knows the true state of the thing. Here, the entrant has only two action: enter or not. That is precisely the two strategies that she can have. The distinction between action and strategy seems trivial, but actually it is really important as I will illustrate in another example below. But for the simple entrant model, if since the incumbent can observe the action of the entrant, and each action precisely correspond with an strategy of the entrant, observing the action if the incumbent is exactly the same as observing the strategy of her. Thus, the only belief formation function for the incumbent is that: the entrant's strategy is the one correspond to the entrant's observed action. There is absolutely no ambiguity here. Such simplicity lead many people into believe in the SPNE but ignored the important detailed assumption about the belief formation function and joint deviation.

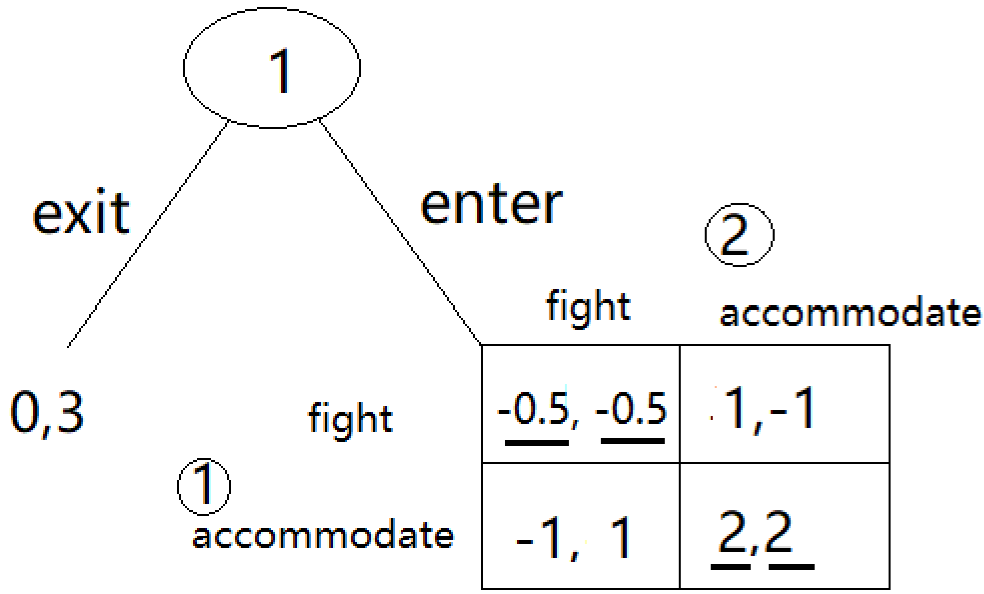


Figure 2: Example of joint deviation

Now, let us look as the following example as in 2. This has one more step than the classical entry model example. There is a third stage for after the incumbent choose to accommodate. The entrant can choose to also accommodate and share the market with the incumbent, but also, she can fight back and gets a payoff equal to (1,-1) if the opponent accommodates. Lastly, if both There are two SPNEs: Player 1 chooses $\{exit, fight\}$ and player 2 chooses $\{fight\}$. The second equilibrium is player 1 chooses $\{exit, accommodate\}$ and player 2 chooses $\{accommodate\}$ In this case, player 1 gets zero and player 2 gets three. Player 1 would be better off if she can commit to accommodate on the third stage. The added stage impose additional challenge to player 2 here. Now, he does not perfectly observe the strategy of her opponent anymore, because the opponent have an additional stage of move that is not observable to him. Then, we can see if the arguments for SPNE still holds.

Assume before the game start, the players are assigned the strategy profile: player 1 chooses $\{exit, fight\}$, player 2 chooses $\{fight\}$. Still, we assume that each players can perfectly observe the history of action on each node. But on the second period, both players move simultaneously and thus cannot observe each other's action of choice.

Now, let us see what would happen if player 1 deviates to *enter* on the first stage. Upon observing such action, player 2 knows that his opponent has abandoned the assigned strategy and set off to an off equilibrium play. But player 2 does not observe the actual strategy of player 1, and thus he has to form an belief about the opponent's strategy. He can still believe that player 1 would choose fight in the second stage, and the best response is to fight too. He can also belief that player 1 choose to enter must because she wants to seek an better payoff than 0, so player 1 would also deviate to accommodate in the second period. In this way, both players can get payoff equal to 2. This is the logic of forward induction.

It is even possible to add a third player in the second stage and play a similar coordination game. Assume still, fight is the best response to everyone fights, and accommodate is the best response to everyone else accommodate. Set the accommodation equilibrium to be the Pareto optimal outcome. Also, assume there is no other Nash equilibrium in the second stage. When player 1 deviates from exit to enter on the first stage, player 2 has to anticipate not only whether player 1 would deviate on the second stage, but also he has to form belief about whether player 3 would also deviate from fight. Since player 3 does not initiate the deviation, and has not moved before. It is unclear how player 2 would form a belief about player 3. But nevertheless, it is possible to build different belief formation functions to predict that would happen using each belief formation function, and we can empirically test what are the belief formation function that people actually use in an empirical way, and theory might not have a decisive answer to those realistic questions.

I think it is likely that the belief formation function have the following structure in general: if there is a deviation from the original equilibrium path and if there is a unique Pareto optimal outcome, then deviate to that outcome. If there are multiple Pareto optimal outcomes, then we have problem.

Sequential contact might also work very well when there is a unique Pareto optimal outcome. But it has problem if there are everyone has the contact information of everyone else. It also have problem when there are multiple Pareto optimal outcomes.

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A Solve the coupon game for the bar owners (unexpected payoff changes)

A.1 Case 1. cost of coupon is between 1 and 1.5

First, we can eliminate the set of strictly dominated strategy for each bar owner given the believes of their customers. Here are some simple lemmas:

1. A bar giving all three customers coupons is a strictly dominated by not giving any coupon. Because in this case, for each customer going to the bar, the owner earn $1/(1-r)$ unit of profit while have to lose $1.4/(1-r)$ unit profit as the cost of free drink. $(-(3 \cdot 64 - 3)$ cases)

2. Bar B giving coupon to only one customer is weakly dominated by B not giving any coupon. Because given the believes, one customer would attract at most one customer, and thus the cost does not cover the benefit. Only when no one comes to the bar B, then there is no further loss. (This conclusion also applies to bar C)

3. It can never be a equilibrium in which all consumers receive at least one coupon. Because a coupon is only profitable when it can attract all three customers to the store. When each consumer revives at least one coupon, it either they all going to one bar, which is not optimal, or they do not go to the same bar in which all the stores have at least one consumer will suffer a loss.

4. The only profitable case of coupon offering is: offer 1 coupon and attract 2 consumers (earn a profit of $\frac{0.6}{1-r}$); offer 2 coupons and attract 3 consumers (earn a profit of $\frac{0.2}{1-r}$) .

B offers coupon to two players cannot be a NE Bar B offers coupon to 2 and 3, and the other two bars does not give away coupon is not a NE. In this case all three players will go to B and B earn a profit $\pi_b = \frac{3-2.8}{1-r}$. Both A and C get zero. B has no profitable deviation, because if he offers less coupon, then customers who does not get the coupon will not come to B and then B suffers a loss. Giving all consumers coupons is also not a good idea, because it is strictly dominated. Giving no consumer coupon would make B worse off too, because the profit reduced.

If A gives player 1 a coupon. Then A suffer a loss, because this policy is not going to attract 2 and 3 back to it. If A gives 3 a coupon, then 3 is indifferent between going to A or B. By our assumption of believe, then player 3 will go to A, so that A can earn a positive profit and be better off.

By similar reason, B offers coupon to 12, 13, while others do not offer coupons also cannot be a equilibrium. Thus B offer coupon to two players cannot be a NE. By symmetry, this also applies to C.

A(2,3), B(3) C(2) A gives both 2 and 3 a coupon. B gives 3 a coupon, C gives 2 a coupon. This is a NE for the bar owners. In this case, all players will go to bar A. Consequently, bar B and C earn zero profit. Bar A, on the other hand, ears a positive profit equals to $\frac{3-2.8}{1-r}$. If A offers less coupon, for instance, A change to only give coupon to 2. Then player 3 will switch to B. Player 2 stays in A, and so does player 1. A earn a profit of $\frac{2-1.4}{1-r}$ which is larger.

A(2,3) B(2,3) C(2): In this case, all players remain at A. If A deviates to A(2), then player 3 will go to B. Player 2 will go to B, because B is a favorite place. then player 1 will follow. So the deviation will not be profitable. If A deviate to A(3), Then 2 will go to B or C. Player 1 and 3 will remain in A. Thus profitable deviation.

A(2,3) B(2,3) C(2,3): In this case, all players remain at A. If A deviates to A(2), Then player 3 goes to B or C with equal prob. Player 1 will not deviate, and so does player 2. Thus, profitable deviation.

A(2,3) B(2,3) C(3):

A(3), B(3), C(0). A gives player 3 a coupon. B gives player 3 a coupon, C does not give a coupon. In this case, all consumers go to bar A. Both B and C earns zero profit. If A stop giving the coupon, then player 3 would go to B, and it would be strictly better off for player 2 to go to B too, because B is the favorite bar for player 2 and she gets an additional unit of utility by going there. Then, going to A is no longer an equilibrium for player 1. So, all of the players will end up in B, and A drop from earning a positive profit to earning a zero profit.

If B gives both player 1 and 3 a coupon. Then all the players will end up in B, so B can do better in this deviation.

A(1) B(2) C(3): Deviate to no coupon.

If bar A does not offer enough coupon to attract any customer, then it must be the case that one of the other two bars offers at most two coupons to attract all three consumers. However, A would be better off giving the coupon to the same two consumers, and then they will all end up in A due to the believes of the players, A will be better off with such deviation. So, if A does not have a customer in the inner NE, the strategy cannot be an NE in the outer game.

If bar A only attract one consumer, it is never going to be an equilibrium too. Because if A use one coupon to attract one consumer, then, it would be better off not giving away a coupon. If A does not offer a coupon, it can never be the case that A gets exactly one consumer. Because consumers would be better off staying together.

If A does not offer coupon, but attracts to attract 2 consumers, it must be the case that the remaining consumer is attracted by another bar j by one coupon. But this cannot be a NE in the outer game, because bar j would be better off offering two coupons and attract all the consumers.

If bar A offers one coupon to attract 2 customers, then the remaining consumer must be attracted by a coupon offered by another bar j, otherwise that consumer will not have incentive to say alone. But this cannot be an equilibrium in the outer game, since the bar j would be better off offering two coupons, one for his current customer and the other one for the customer who gets the coupon from A.

Bar A will not offer two or more coupons to attract 2 consumers, because in such case, it will lose profit.

If Bar A does not offer any coupon, but have all three consumers, this is only true when the other two bars does not offer any coupon. Then another bar j would be strictly better off giving away two coupons and attract all consumers.

Also, in a NE of the outer game, the bar A cannot use one coupon to attract three consumers. Because the one of the other two bars j can offer a coupon to the one player who gets the coupon from A and another coupon to another player who does not initially get a coupon from A. With this offer, all consumers will go to j.

Lastly, we consider the case in which A offers two coupons to attract 3 consumers. This is the case where we will find pure strategy NE. When A offers two coupons, then all the other bars will offer at most two coupons to the same two consumers, and there is always one customer who does not get the coupon (by lemma 3).

The set of equilibria is: Bar A offers coupon to exactly two players, and at least one of the other bar offers coupon to the same two players and there the remaining player does not get coupon from any bar. This is a NE because if A reduces the number of coupons, then due to the believes of the players, they will go to the other bar who offers two coupons. In this case, A would earn less profit. Also, due to lemma 3, it is never optimal for A to offer coupons to all the player. So bar A does not have any profitable deviation. Then, we check the B and C does not have profitable deviation either. Again, due to lemma 3, neither of them will deviate to offer coupons to all the players. Then, as long as they offer coupon to the same two players as A, still, all the players will go to A because their believe in equilibrium selection. So even if B and C reduces the number of coupons, they still earn a profit of zero. Last, we need to consider if it would be optimal for B or C switch to offer coupons to two players, one of which bar A does not offer an coupon (see the example below). This is never going to be a profitable deviation, because the deviater would at best attract 2 consumers with 2 coupons. This is going to be a loss.

Example A(2,3), B(2,3), C(2,3), is it an equilibrium. All players will be in A and only A would earn a positive profit. If B switch to (1,2), then 1 would go to B, 2 would prefer B to A, 3 would not go to B, because A and C still offers coupons to 3. In this case, B would lose money, because she uses two coupons and attract only two players. It is also not optimal for B to choose B(1), because it is not profitable to attract one consumer with one coupon. By symmetry, there will not be profitable deviation for C either. So A(2,3), B(2,3), C(2,3) is an equilibrium.

Example A(1,2), B(1,2), C(1,2), if A get rid of 1, then 1 will go to bar B. Then 2 will follow 1, as she prefers B. So this is not a profitable deviation. If A get rid of 2, then 2 will go to B. Then B_1 B_2 B_3 Pareto dominates A_1 B_2 A_3 (player 1 is indifferent, but player 2 and 3 are strictly better off)

A.2 If Cost of coupon > 1.5 & < 2

, so the only profitable coupon offering will be one coupon attracts two consumers

A.3 If cost of coupon < 1

Resulting equilibrium is that all stores give coupons to all consumers. Yet, all consumers still stay at A.

A.4 If cost of coupon > 2 & < 3

Have to use one coupon to attract tree consumers, it is never optimal to offer two coupons.

A.5 If cost of coupon > 3

Then it is never optimal to offer coupons.

B Appendix: Naive probability belief against myopic best response